Consider the following eigenstates of a hypothetical quantum system.

$$
\begin{aligned}
|000\rangle &= (1\ 0\ 0\ 0\ 0\ 0\ 0)^\dagger && \text{no fermions} \\
|100\rangle &= (0\ 1\ 0\ 0\ 0\ 0\ 0)^\dagger && \text{one fermion in state } \phi_1 \\
|010\rangle &= (0\ 0\ 1\ 0\ 0\ 0\ 0)^\dagger && \text{one fermion in state } \phi_2 \\
|001\rangle &= (0\ 0\ 0\ 1\ 0\ 0\ 0)^\dagger && \text{one fermion in state } \phi_3 \\
|110\rangle &= (0\ 0\ 0\ 0\ 1\ 0\ 0)^\dagger && \text{two fermions, one in state } \phi_1\text{, one in state } \phi_2 \\
|101\rangle &= (0\ 0\ 0\ 0\ 0\ 1\ 0)^\dagger && \text{two fermions, one in state } \phi_1\text{, one in state } \phi_3 \\
|011\rangle &= (0\ 0\ 0\ 0\ 0\ 0\ 1)^\dagger && \text{two fermions, one in state } \phi_2\text{, one in state } \phi_3
\end{aligned}
$$

Let fermion states ϕ_n be modeled by a one dimensional box of length L.

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Fermion creation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$$
\begin{aligned}
\hat{b}_1^\dagger &= |100\rangle\langle 000| - |110\rangle\langle 010| - |101\rangle\langle 001| && \text{Create one fermion in state } \phi_1 \\
\hat{b}_2^\dagger &= |010\rangle\langle 000| + |110\rangle\langle 100| - |011\rangle\langle 001| && \text{Create one fermion in state } \phi_2 \\
\hat{b}_3^\dagger &= |001\rangle\langle 000| + |101\rangle\langle 100| + |011\rangle\langle 010| && \text{Create one fermion in state } \phi_3
\end{aligned}
$$

Fermion annihilation operators are the adjoint of creation operators.

$$
\begin{aligned}
\hat{b}_1 &= (\hat{b}_1^\dagger)^\dagger && \text{Annihilate one fermion in state } \phi_1 \\
\hat{b}_2 &= (\hat{b}_2^\dagger)^\dagger && \text{Annihilate one fermion in state } \phi_2 \\
\hat{b}_3 &= (\hat{b}_3^\dagger)^\dagger && \text{Annihilate one fermion in state } \phi_3
\end{aligned}
$$

Given the wavefunction operator

$$\hat{\psi} = \frac{1}{\sqrt{2}}\sum_{n,m}\phi_n(x)\phi_m(y)\hat{b}_n\hat{b}_m$$

show that

$$\hat{\psi}|110\rangle = \frac{1}{\sqrt{2}}\big(\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x)\big)|000\rangle$$

$$\hat{\psi}|101\rangle = \frac{1}{\sqrt{2}}\big(\phi_1(x)\phi_3(y) - \phi_1(y)\phi_3(x)\big)|000\rangle$$

$$\hat{\psi}|011\rangle = \frac{1}{\sqrt{2}}\big(\phi_2(x)\phi_3(y) - \phi_2(y)\phi_3(x)\big)|000\rangle$$