The electric and magnetic field vectors can be derived from a vector potential. For example, the following vector potential represents a light wave moving in the z direction.

$$\mathbf{A} = \begin{pmatrix} -E_x\omega^{-1}\cos(kz-\omega t) \\ E_y\omega^{-1}\sin(kz-\omega t) \\ 0 \end{pmatrix}$$

where $k = \omega/c$. The symbols E_x and E_y are used for component magnitudes to avoid confusion with the eigenvalues A_x and A_y that will appear soon. The corresponding electric and magnetic field vectors are

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} = \begin{pmatrix} E_x\sin(kz-\omega t) \\ E_y\cos(kz-\omega t) \\ 0 \end{pmatrix} \qquad \mathbf{B} = c\nabla\times\mathbf{A} = \begin{pmatrix} -E_y\cos(kz-\omega t) \\ E_x\sin(kz-\omega t) \\ 0 \end{pmatrix}$$

The photon wave function for $\mathbf{A}$ can be written as

$$\psi = (n_x!n_y!)^{-1/2}(a_x^\dagger)^{n_x}(a_y^\dagger)^{n_y}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix} \exp(in_x(kz-\omega t+\phi_x)) \\ \exp(in_y(kz-\omega t+\phi_y)) \\ 1 \end{pmatrix}$$

where the integers n_x and n_y are numbers of photons. The vector (1,1,1) is the vacuum state $|0\rangle$ with no photons. The photon annihilation and creation operators are

$$a_x = \sqrt{n_x}\begin{pmatrix} \exp(-i(kz-\omega t+\phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad a_x^\dagger = \sqrt{n_x+1}\begin{pmatrix} \exp(i(kz-\omega t+\phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_y = \sqrt{n_y}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(-i(kz-\omega t+\phi_y)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad a_y^\dagger = \sqrt{n_y+1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i(kz-\omega t+\phi_y)) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_z = \sqrt{n_z}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-i(kz-\omega t+\phi_z)) \end{pmatrix} \qquad a_z^\dagger = \sqrt{n_z+1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(i(kz-\omega t+\phi_z)) \end{pmatrix}$$

The number of photons can be computed from an arbitrary ψ as follows.

$$\sqrt{n_x}\psi_x = \left(i\omega^{-1}\psi_x\frac{\partial}{\partial t}\psi_x\right)^{1/2} \qquad \sqrt{n_x+1}\psi_x = \left(i\omega^{-1}\psi_x\frac{\partial}{\partial t}\psi_x + \psi_x^2\right)^{1/2}$$

$$\sqrt{n_y}\psi_y = \left(i\omega^{-1}\psi_y\frac{\partial}{\partial t}\psi_y\right)^{1/2} \qquad \sqrt{n_y+1}\psi_y = \left(i\omega^{-1}\psi_y\frac{\partial}{\partial t}\psi_y + \psi_y^2\right)^{1/2}$$

$$\sqrt{n_z}\psi_z = \left(i\omega^{-1}\psi_z\frac{\partial}{\partial t}\psi_z\right)^{1/2} \qquad \sqrt{n_z+1}\psi_z = \left(i\omega^{-1}\psi_z\frac{\partial}{\partial t}\psi_z + \psi_z^2\right)^{1/2}$$

The vector potential operators are

$$\hat{A}_x = iC\omega^{-1}(a_x - a_x^\dagger) \qquad \hat{A}_y = iC\omega^{-1}(a_y - a_y^\dagger) \qquad \hat{A}_z = iC\omega^{-1}(a_z - a_z^\dagger)$$

where C is a conversion constant. Recall that the eigenvalues of operators are observables. Let us now compute the eigenvalue A_x.

$$\begin{aligned}
\hat{A}_x\psi &= iC\omega^{-1}(a_x\psi - a_x^\dagger\psi) \\
&= iC\omega^{-1}\sqrt{n_x}\begin{pmatrix}\exp(-i(kz-\omega t+\phi_x)) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\psi - iC\omega^{-1}\sqrt{n_x+1}\begin{pmatrix}\exp(i(kz-\omega t+\phi_x)) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\psi \\
&= A_x\psi
\end{aligned}$$

Hence the eigenvalue is

$$A_x = iC\omega^{-1}\sqrt{n_x}\begin{pmatrix}\exp(-i(kz-\omega t+\phi_x)) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - iC\omega^{-1}\sqrt{n_x+1}\begin{pmatrix}\exp(i(kz-\omega t+\phi_x)) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

For large n_x the approximation $\sqrt{n_x+1} \approx \sqrt{n_x}$ yields

$$A_x = iC\omega^{-1}\sqrt{n_x}\Big[\exp(-i(kz-\omega t+\phi_x)) - \exp(i(kz-\omega t+\phi_x))\Big]\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Recall that $2\sin(\alpha) = i[\exp(-i\alpha) - \exp(i\alpha)]$ hence

$$A_x = 2C\omega^{-1}\sqrt{n_x}\sin(kz-\omega t+\phi_x)\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

A similar calculation for the y direction yields

$$A_y = 2C\omega^{-1}\sqrt{n_y}\sin(kz-\omega t+\phi_y)\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

For the z direction the wave function ψ has $n_z = 0$ photons hence

$$\begin{aligned}
\hat{A}_z\psi &= iC\omega^{-1}(a_z\psi - a_z^\dagger\psi) \\
&= iC\omega^{-1}\sqrt{0}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \exp(-i(kz-\omega t+\phi_z))\end{pmatrix}\psi - iC\omega^{-1}\sqrt{1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \exp(i(kz-\omega t+\phi_z))\end{pmatrix}\psi \\
&= A_z\psi
\end{aligned}$$

Note that A_z is an eigenvalue of a single photon and the contribution of A_z to the magnitude of $\mathbf{A}$ essentially vanishes for large n_x or n_y. Hence the A_z shown above is discarded and the approximation $A_z = 0$ is used instead. Putting all the eigenvalues together we have

$$\mathbf{A} = A_x\begin{pmatrix}1\\0\\0\end{pmatrix} + A_y\begin{pmatrix}0\\1\\0\end{pmatrix} + A_z\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2C\omega^{-1}\sqrt{n_x}\sin(kz-\omega t+\phi_x)\\ 2C\omega^{-1}\sqrt{n_y}\sin(kz-\omega t+\phi_y)\\ 0\end{pmatrix}$$

Choosing $\phi_x = -\pi/2$ and $\phi_y = 0$ we obtain

$$\mathbf{A} = \begin{pmatrix}-2C\omega^{-1}\sqrt{n_x}\cos(kz-\omega t)\\ 2C\omega^{-1}\sqrt{n_y}\sin(kz-\omega t)\\ 0\end{pmatrix}$$

This is equivalent to the original classical $\mathbf{A}$ with $E_x = 2C\sqrt{n_x}$ and $E_y = 2C\sqrt{n_y}$.

```
-- www.eigenmath.org/vector-potential.txt
-- See also www.calctool.org/CALC/chem/photochemistry/power_photons
nx = 2517 * 10^15 -- 1 W at 500 nm
ny = 5034 * 10^15 -- 2 W at 500 nm
GX = exp(i*(k*z - omega*t + phix))
GY = exp(i*(k*z - omega*t + phiy))
psi = (GX^nx, GY^ny, 0)
N(j) = sqrt(i / omega * psi[j] * d(psi[j],t))
N1(j) = sqrt(i / omega * psi[j] * d(psi[j],t) + psi[j]^2)
AX = i * C/omega * (conj(GX) * N(1) - GX * N1(1))
AY = i * C/omega * (conj(GY) * N(2) - GY * N1(2))
-- divide by psi to get eigenvalues
AX = AX / psi[1]
AY = AY / psi[2]
-- large n approx
AX = subst(-i*C/omega*sqrt(nx)*GX, -i*C/omega*sqrt(nx+1)*GX, AX)
AY = subst(-i*C/omega*sqrt(ny)*GY, -i*C/omega*sqrt(ny+1)*GY, AY)
"verify A (0 = ok)"
phix = -pi/2
phiy = 0
AX + 2 * C/omega * sqrt(nx) * expcos(k*z - omega*t)
AY - 2 * C/omega * sqrt(ny) * expsin(k*z - omega*t)
"check maxwell equations (0 = ok)"
k = omega / c
A = (AX, AY, 0)
E = -d(A,t)
B = c * curl(A)
div(E)
div(B)
curl(E) + d(B,t)/c
curl(B) - d(E,t)/c
```