Recall that the following wave functions are solutions to the Dirac equation.

$$\psi_1 = \begin{pmatrix} \omega + m \\ 0 \\ k_z \\ k_x + ik_y \end{pmatrix} \exp[i(k_x x + k_y y + k_z z - \omega t)] \quad \psi_7 = \begin{pmatrix} k_z \\ k_x + ik_y \\ \omega + m \\ 0 \end{pmatrix} \exp[-i(k_x x + k_y y + k_z z - \omega t)]$$

$$\psi_2 = \begin{pmatrix} 0 \\ \omega + m \\ k_x - ik_y \\ -k_z \end{pmatrix} \exp[i(k_x x + k_y y + k_z z - \omega t)] \quad \psi_8 = \begin{pmatrix} k_x - ik_y \\ -k_z \\ 0 \\ \omega + m \end{pmatrix} \exp[-i(k_x x + k_y y + k_z z - \omega t)]$$

A spinor is the vector part of a Dirac wave function. The following eight spinors are used for scattering calculations. The u spinors are fermions from ψ_1 and ψ_2. The v spinors are anti-fermions from ψ_7 and ψ_8. The last digit of the u or v subscript is 1 for spin up and 2 for spin down.

$$u_{11} = \begin{pmatrix} E_1 + m_1 \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix} \quad v_{21} = \begin{pmatrix} p_{2z} \\ p_{2x} + ip_{2y} \\ E_2 + m_2 \\ 0 \end{pmatrix} \quad u_{31} = \begin{pmatrix} E_3 + m_3 \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix} \quad v_{41} = \begin{pmatrix} p_{4z} \\ p_{4x} + ip_{4y} \\ E_4 + m_4 \\ 0 \end{pmatrix}$$

$$u_{12} = \begin{pmatrix} 0 \\ E_1 + m_1 \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix} \quad v_{22} = \begin{pmatrix} p_{2x} - ip_{2y} \\ -p_{2z} \\ 0 \\ E_2 + m_2 \end{pmatrix} \quad u_{32} = \begin{pmatrix} 0 \\ E_3 + m_3 \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix} \quad v_{42} = \begin{pmatrix} p_{4x} - ip_{4y} \\ -p_{4z} \\ 0 \\ E_4 + m_4 \end{pmatrix}$$

These are the associated momentum vectors.

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} \quad p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} \quad p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix} \quad p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}$$

Spinors are solutions to the following momentum-space Dirac equation with $\not{p} = p \cdot (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$.

$$(\not{p} - m)u = 0 \qquad (\not{p} + m)v = 0$$

Up and down spinors have the following "completeness property."

$$u_{11}\bar{u}_{11} + u_{12}\bar{u}_{12} = (E_1 + m_1)(\not{p}_1 + m_1) \qquad v_{21}\bar{v}_{21} + v_{22}\bar{v}_{22} = (E_2 + m_2)(\not{p}_2 - m_2)$$

The adjoint of a spinor is $\bar{u} = u^\dagger \gamma^0$. The adjoint is a row vector hence $u\bar{u}$ is an outer product. The script below verifies all eight spinors.

```
-- www.eigenmath.org/spinors.txt
E1 = sqrt(p1x^2 + p1y^2 + p1z^2 + m1^2)
E2 = sqrt(p2x^2 + p2y^2 + p2z^2 + m2^2)
E3 = sqrt(p3x^2 + p3y^2 + p3z^2 + m3^2)
E4 = sqrt(p4x^2 + p4y^2 + p4z^2 + m4^2)
```

```
p1 = (E1,p1x,p1y,p1z)
p2 = (E2,p2x,p2y,p2z)
p3 = (E3,p3x,p3y,p3z)
p4 = (E4,p4x,p4y,p4z)
u11 = (E1+m1,0,p1z,p1x+i*p1y)
u12 = (0,E1+m1,p1x-i*p1y,-p1z)
v21 = (p2z,p2x+i*p2y,E2+m2,0)
v22 = (p2x-i*p2y,-p2z,0,E2+m2)
u31 = (E3+m3,0,p3z,p3x+i*p3y)
u32 = (0,E3+m3,p3x-i*p3y,-p3z)
v41 = (p4z,p4x+i*p4y,E4+m4,0)
v42 = (p4x-i*p4y,-p4z,0,E4+m4)
I = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))
gmunu = ((1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma0 = ((1,0,0,0),(0,1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma1 = ((0,0,0,1),(0,0,1,0),(0,-1,0,0),(-1,0,0,0))
gamma2 = ((0,0,0,-i),(0,0,i,0),(0,i,0,0),(-i,0,0,0))
gamma3 = ((0,0,1,0),(0,0,0,-1),(-1,0,0,0),(0,1,0,0))
gamma = (gamma0,gamma1,gamma2,gamma3)
pslash1 = dot(gmunu,p1,gamma)
pslash2 = dot(gmunu,p2,gamma)
pslash3 = dot(gmunu,p3,gamma)
pslash4 = dot(gmunu,p4,gamma)
"dirac equation"
dot(pslash1-m1*I,u11)
dot(pslash1-m1*I,u12)
dot(pslash2+m2*I,v21)
dot(pslash2+m2*I,v22)
dot(pslash3-m3*I,u31)
dot(pslash3-m3*I,u32)
dot(pslash4+m4*I,v41)
dot(pslash4+m4*I,v42)
"completeness"
outer(u11,dot(conj(u11),gamma0))+outer(u12,dot(conj(u12),gamma0))-
  (E1+m1)*(pslash1+m1*I)
outer(v21,dot(conj(v21),gamma0))+outer(v22,dot(conj(v22),gamma0))-
  (E2+m2)*(pslash2-m2*I)
outer(u31,dot(conj(u31),gamma0))+outer(u32,dot(conj(u32),gamma0))-
  (E3+m3)*(pslash3+m3*I)
outer(v41,dot(conj(v41),gamma0))+outer(v42,dot(conj(v42),gamma0))-
  (E4+m4)*(pslash4-m4*I)
```