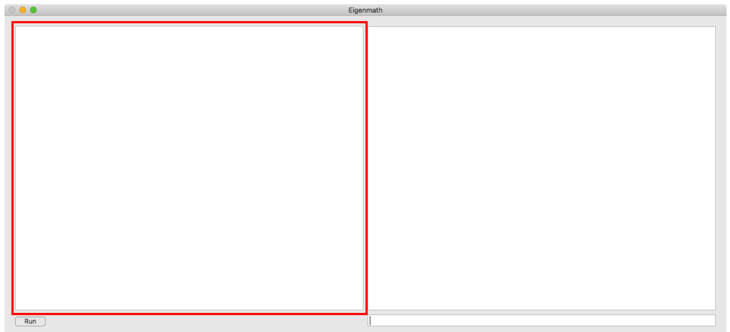
SIMPLE PROBLEMS

Here are some simple but interesting problems with example scripts. The scripts, shown in blue color, can be pasted into the Eigenmath script window and then run by clicking the Run button.



Let

$$\mathbf{E} = \begin{pmatrix} A\sin(kz - \omega t + \phi) \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 \\ A\sin(kz - \omega t + \phi) \\ 0 \end{pmatrix}$$

where $k = \omega/c$. Verify that $\mathbf{E}$ and $\mathbf{B}$ are solutions to the free-field Maxwell equations

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + c^{-1}\frac{\partial}{\partial t}\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\nabla \times \mathbf{B} - c^{-1}\frac{\partial}{\partial t}\mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
k = omega/c
E = (A*sin(k*z - omega*t + phi), 0, 0)
B = (0, A*sin(k*z - omega*t + phi), 0)
div(E)
div(B)
curl(E) + d(B,t)/c
curl(B) - d(E,t)/c
```

Let

$$\psi = \exp(ik_x x + ik_y y + ik_z z - i\omega t)$$

where

$$\omega = \sqrt{k_x^2 + k_y^2 + k_z^2 + m^2}$$

Verify that ψ is a solution to the Klein-Gordon equation

$$\frac{\partial^2}{\partial t^2}\psi - \frac{\partial^2}{\partial x^2}\psi - \frac{\partial^2}{\partial y^2}\psi - \frac{\partial^2}{\partial z^2}\psi + m^2\psi = 0$$

```
omega = sqrt(kx^2 + ky^2 + kz^2 + m^2)
psi = exp(i*kx*x + i*ky*y + i*kz*z - i*omega*t)
d(psi,t,t) - d(psi,x,x) - d(psi,y,y) - d(psi,z,z) + m^2*psi
```

Verify the following propagator identity for a non-relativistic free particle moving in one dimension.

$$\psi(y,t) = \int_{-\infty}^{\infty} G(y,x,t)\psi(x,0)\,dx$$

The propagator is

$$G(y,x,t) = \sqrt{-\frac{im}{2\pi t}}\exp\left(\frac{im(y-x)^2}{2t}\right)$$

The non-relativistic wave function is

$$\psi(x,t) = \exp(ipx - iEt)$$

with energy

$$E = \frac{p^2}{2m}$$

Start by expanding the integrand.

$$G(y,x,t)\psi(x,0) = \sqrt{-\frac{im}{2\pi t}}\exp\left(\frac{imy^2}{2t}\right)\exp\left(-\frac{imyx}{t}\right)\exp\left(\frac{imx^2}{2t}\right)\exp(ipx)$$

Partition terms not involving x.

$$\psi(y,t) = \sqrt{-\frac{im}{2\pi t}}\exp\left(\frac{imy^2}{2t}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{imyx}{t}\right)\exp\left(\frac{imx^2}{2t}\right)\exp(ipx)\,dx$$

Use the Gaussian integral

$$\int_{-\infty}^{\infty}\exp\left(\frac{1}{2}ax^2 + bx\right)\,dx = \sqrt{\frac{2\pi}{a}}\exp\left(\frac{b^2}{2a}\right)$$

with

$$a = -\frac{im}{t} \qquad b = -\frac{imy}{t} + ip$$

to obtain

$$\psi(y,t) = \sqrt{-\frac{im}{2\pi t}}\exp\left(\frac{imy^2}{2t}\right)\sqrt{-\frac{2\pi t}{im}}\underbrace{\exp\left(-\frac{imy^2}{2t} + ipy - \frac{ip^2 t}{2m}\right)}_{\exp\left(\frac{b^2}{2a}\right)}$$

$$= \exp(ipy - iEt)$$

```
E = p^2/(2*m)
psi(x,t) = exp(i*p*x - i*E*t)
a = -i*m/t
b = -i*m*y/t + i*p
exp(i*m*y^2/(2*t)) * exp(b^2/(2*a))
```

Verify the following equation.

$$\vec{k}\cdot\vec{\sigma}=\begin{pmatrix}k_z & k_x-ik_y\\ k_x+ik_y & -k_z\end{pmatrix}$$

```
sigmax = ((0,1),(1,0))
sigmay = ((0,-i),(i,0))
sigmaz = ((1,0),(0,-1))
sigma = (sigmax,sigmay,sigmaz)
k = (kx,ky,kz)
dot(k,sigma)
```

The dot product is commutative but if you try to compute $\vec{\sigma}\cdot\vec{k}$ with Eigenmath an error occurs. This is because $\vec{\sigma}$ is not really a vector, it's a tensor. To compute $\vec{\sigma}\cdot\vec{k}$ it is necessary to transpose indices.

```
sigmax = ((0,1),(1,0))
sigmay = ((0,-i),(i,0))
sigmaz = ((1,0),(0,-1))
sigma = (sigmax,sigmay,sigmaz)
sigma = transpose(sigma,1,2)
sigma = transpose(sigma,2,3)
k = (kx,ky,kz)
dot(sigma,k)
```

Another method is to contract an outer product.

```
sigmax = ((0,1),(1,0))
sigmay = ((0,-i),(i,0))
sigmaz = ((1,0),(0,-1))
sigma = (sigmax,sigmay,sigmaz)
k = (kx,ky,kz)
T = outer(k,sigma)
contract(T,1,2)
T = outer(sigma,k)
contract(T,1,4)
```