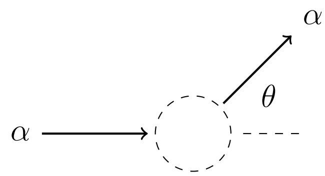
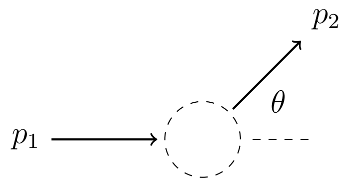
Alpha particles are scattered by atomic nuclei due to the Coulomb force.



This is the same diagram with momentum labels.



These are the momentum vectors for p_1 aligned with the z axis and with $p = \sqrt{E^2 - m^2}$.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}$$

The scattering probability density is $f(\theta,\phi) = \langle|\mathcal{M}|^2\rangle$ where $\mathcal{M}$ is the scattering amplitude, $|\mathcal{M}|^2$ is the scattering probability, and $\langle|\mathcal{M}|^2\rangle$ is the spin average of $|\mathcal{M}|^2$. The scattering probability density is computed from the momentum vectors as follows.

$$\begin{aligned} f(\theta,\phi) &= \frac{C}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\left|\bar{u}_2\gamma^0 u_1\right|^2 \\ &= \frac{C}{4}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^0(\not{p}_2 + m)\gamma^0\right] \\ &= C\left(E^2 + m^2 + p^2\cos\theta\right) \end{aligned}$$

where

$$C = \frac{Z^2 e^4}{8p^4\sin^4\theta/2}$$

The following script verifies that

$$\frac{1}{4}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^0(\not{p}_2 + m)\gamma^0\right] = E^2 + m^2 + p^2\cos\theta$$

```
-- www.eigenmath.org/rutherford.txt
p = sqrt(E^2 - m^2)
p1 = (E,0,0,p)
p2 = (E,
      p*expsin(theta)*expcos(phi),
      p*expsin(theta)*expsin(phi),
      p*expcos(theta))
```

```
I = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))
gmunu = ((1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma0 = ((1,0,0,0),(0,1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma1 = ((0,0,0,1),(0,0,1,0),(0,-1,0,0),(-1,0,0,0))
gamma2 = ((0,0,0,-i),(0,0,i,0),(0,i,0,0),(-i,0,0,0))
gamma3 = ((0,0,1,0),(0,0,0,-1),(-1,0,0,0),(0,1,0,0))
gamma = (gamma0,gamma1,gamma2,gamma3)
pslash1 = dot(gmunu,p1,gamma)
pslash2 = dot(gmunu,p2,gamma)
A = contract(dot(pslash1+m*I,gamma0,pslash2+m*I,gamma0))
B = E^2 + m^2 + p^2*expcos(theta)
-- this difference should be zero
A/4-B
```

The following script shows that

$$\frac{E^2 + m^2 + p^2 \cos\theta}{8p^4} = \frac{1 - \beta^2 \sin^2\theta/2}{4p^2\beta^2}$$

where $p = \sqrt{E^2 - m^2}$ and $\beta = p/E$. The exponential forms of sine and cosine are used so that trigonometric identities are not required.

```
p = sqrt(E^2 - m^2)
beta = p/E
A = (E^2 + m^2 + p^2 * expcos(theta)) / (8*p^4)
B = (1 - beta^2 * expsin(theta/2)^2) / (4*p^2*beta^2)
A-B
```