Vector $\mathbf{E}$ is an electric field vector for elliptically polarized light moving in the z direction.

$$\mathbf{E} = \begin{pmatrix} A_x \sin(kz - \omega t) \\ A_y \cos(kz - \omega t) \\ 0 \end{pmatrix}$$

Here is an example showing the rotation of $\mathbf{E}$. Let $z = 0$ and $\omega = 2\pi$, and let the amplitudes A_x and A_y be positive. Then the vector $\mathbf{E}$ points to twelve o'clock at time $t = 0$ and rotates counter-clockwise to nine o'clock at time $t = 1/4$.

$$\begin{matrix} \begin{pmatrix} 0 \\ A_y \\ 0 \end{pmatrix} & \frac{1}{\sqrt{2}} \begin{pmatrix} -A_x \\ A_y \\ 0 \end{pmatrix} & \begin{pmatrix} -A_x \\ 0 \\ 0 \end{pmatrix} \\ \\ \uparrow & \nwarrow & \longleftarrow \\ \\ t = 0 & t = 1/8 & t = 1/4 \end{matrix}$$

If $A_x = 0$ or $A_y = 0$ then the light is linearly polarized. If $A_x = A_y$ then the light is circularly polarized. Otherwise, the light is elliptically polarized. Shifting now to photons, this is the corresponding photon wave function for $\mathbf{E}$.

$$\psi = (n_x! n_y!)^{-1/2} (a_x^\dagger)^{n_x} (a_y^\dagger)^{n_y} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \exp(i n_x (kz - \omega t + \phi_x)) \\ \exp(i n_y (kz - \omega t + \phi_y)) \\ 1 \end{pmatrix}$$

The integers n_x and n_y are numbers of photons. The vector $(1, 1, 1)$ is the vacuum state $|0\rangle$ with no photons. The photon annihilation and creation operators are

$$a_x = \sqrt{n_x} \begin{pmatrix} \exp(-i(kz - \omega t + \phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad a_x^\dagger = \sqrt{n_x + 1} \begin{pmatrix} \exp(i(kz - \omega t + \phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_y = \sqrt{n_y} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(-i(kz - \omega t + \phi_y)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad a_y^\dagger = \sqrt{n_y + 1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i(kz - \omega t + \phi_y)) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_z = \sqrt{n_z} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-i(kz - \omega t + \phi_z)) \end{pmatrix} \qquad a_z^\dagger = \sqrt{n_z + 1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(i(kz - \omega t + \phi_z)) \end{pmatrix}$$

The number of photons can be computed from an arbitrary ψ as follows.

$$\sqrt{n_x}\psi_x = \left(i\omega^{-1} \psi_x \frac{\partial}{\partial t} \psi_x \right)^{1/2} \qquad \sqrt{n_x + 1}\psi_x = \left(i\omega^{-1} \psi_x \frac{\partial}{\partial t} \psi_x + \psi_x^2 \right)^{1/2}$$

$$\sqrt{n_y}\psi_y = \left(i\omega^{-1} \psi_y \frac{\partial}{\partial t} \psi_y \right)^{1/2} \qquad \sqrt{n_y + 1}\psi_y = \left(i\omega^{-1} \psi_y \frac{\partial}{\partial t} \psi_y + \psi_y^2 \right)^{1/2}$$

$$\sqrt{n_z}\psi_z = \left(i\omega^{-1} \psi_z \frac{\partial}{\partial t} \psi_z \right)^{1/2} \qquad \sqrt{n_z + 1}\psi_z = \left(i\omega^{-1} \psi_z \frac{\partial}{\partial t} \psi_z + \psi_z^2 \right)^{1/2}$$

The electric field operators are

$$\hat{A}_x = iC(a_x - a_x^\dagger) \qquad \hat{A}_y = iC(a_y - a_y^\dagger) \qquad \hat{A}_z = iC(a_z - a_z^\dagger)$$

Apply $\hat{E}_x$ to ψ to obtain

$$\begin{aligned}
\hat{E}_x\psi &= iC(a_x\psi - a_x^\dagger\psi) \\
&= iC\sqrt{n_x}\begin{pmatrix}\exp(-i(kz-\omega t+\phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\psi - iC\sqrt{n_x+1}\begin{pmatrix}\exp(i(kz-\omega t+\phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\psi \\
&= E_x\psi
\end{aligned}$$

Hence the eigenvalue is

$$E_x = iC\sqrt{n_x}\begin{pmatrix}\exp(-i(kz-\omega t+\phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} - iC\sqrt{n_x+1}\begin{pmatrix}\exp(i(kz-\omega t+\phi_x)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}$$

For large n_x the approximation $\sqrt{n_x+1} \approx \sqrt{n_x}$ yields

$$E_x = 2C\sqrt{n_x}\sin(kz-\omega t+\phi_x)\begin{pmatrix}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$$

Similarly

$$E_y = 2C\sqrt{n_y}\sin(kz-\omega t+\phi_y)\begin{pmatrix}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{pmatrix}$$

For the z direction the wave function ψ has $n_z = 0$ photons hence

$$\begin{aligned}
\hat{E}_z\psi &= iC(a_z\psi - a_z^\dagger\psi) \\
&= iC\sqrt{0}\begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-i(kz-\omega t+\phi_z))\end{pmatrix}\psi - iC\sqrt{1}\begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(i(kz-\omega t+\phi_z))\end{pmatrix}\psi \\
&= E_z\psi
\end{aligned}$$

Note that E_z is an eigenvalue of a single photon and the contribution of E_z to the magnitude of $\mathbf{E}$ essentially vanishes for large n_x or n_y. Hence the E_z shown above is discarded and the approximation $E_z = 0$ is used instead. To summarize, we have

$$\begin{aligned}
E_x &= 2C\sqrt{n_x}\sin(kz-\omega t+\phi_x)\begin{pmatrix}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \\
E_y &= 2C\sqrt{n_y}\sin(kz-\omega t+\phi_y)\begin{pmatrix}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{pmatrix} \\
E_z &= 0
\end{aligned}$$

The electric field vector is

$$\mathbf{E} = E_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + E_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + E_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2C\sqrt{n_x}\sin(kz - \omega t + \phi_x) \\ 2C\sqrt{n_y}\sin(kz - \omega t + \phi_y) \\ 0 \end{pmatrix}$$

With $\phi_x = 0$ and $\phi_y = \pi/2$ we have

$$\mathbf{E} = \begin{pmatrix} 2C\sqrt{n_x}\sin(kz - \omega t) \\ 2C\sqrt{n_y}\cos(kz - \omega t) \\ 0 \end{pmatrix}$$

This is equivalent to the original $\mathbf{E}$ with $A_x = 2C\sqrt{n_x}$ and $A_y = 2C\sqrt{n_y}$.

```
-- www.eigenmath.org/polarized-light.txt
nx = 1000
ny = 2000
GX = exp(i*(k*z - omega*t + phix))
GY = exp(i*(k*z - omega*t + phiy))
psi = (GX^nx, GY^ny, 0)
N(j) = sqrt(i / omega * psi[j] * d(psi[j],t))
N1(j) = sqrt(i / omega * psi[j] * d(psi[j],t) + psi[j]^2)
EX = i * C * (conj(GX) * N(1) - GX * N1(1))
EY = i * C * (conj(GY) * N(2) - GY * N1(2))
-- divide by psi to get eigenvalues
EX = EX / psi[1]
EY = EY / psi[2]
-- large n approx
EX = subst(-i*C*sqrt(nx)*GX, -i*C*sqrt(nx+1)*GX, EX)
EY = subst(-i*C*sqrt(ny)*GY, -i*C*sqrt(ny+1)*GY, EY)
"verify E (0 = ok)"
phix = 0
phiy = pi/2
EX - 2 * C * sqrt(nx) * expsin(k*z - omega*t)
EY - 2 * C * sqrt(ny) * expcos(k*z - omega*t)
```