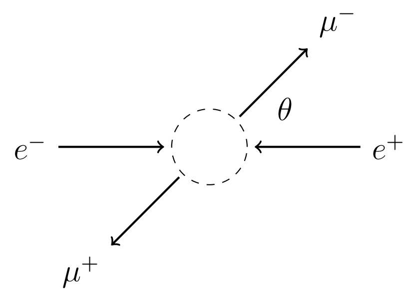
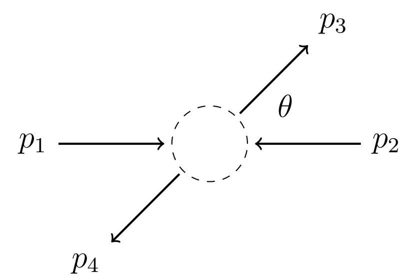
A high energy electron and positron collision can create two muons.



This is the same diagram with momentum labels.



These are the momentum vectors for a center-of-mass (i.e., laboratory) frame of reference with the electron and positron beams aligned with the z axis.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ \rho \sin\theta \cos\phi \\ \rho \sin\theta \sin\phi \\ \rho\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -\rho \sin\theta \cos\phi \\ -\rho \sin\theta \sin\phi \\ -\rho\cos\theta \end{pmatrix}$$

Here $p = \sqrt{E^2 - m^2}$ and $\rho = \sqrt{E^2 - M^2}$ with electron mass $m = 0.51$ MeV and muon mass $M = 106$ MeV. Cross-section calculations require the expectation value $\overline{\mathscr{M}^2}$ which is computed from the momentum vectors as follows.

$$\overline{\mathscr{M}^2} = \frac{e^4}{64E^4} \operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 - m)\gamma^\nu\right] \operatorname{Tr}\left[(\not{p}_4 - M)\gamma_\mu(\not{p}_3 + M)\gamma_\nu\right]$$

The result simplifies to the following expression.

$$\overline{\mathscr{M}^2} = e^4\left[1 + \cos^2\theta + \frac{m^2 + M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta\right]$$

The Stanford Linear Collider had a collision energy of $2E = 91$ GeV. For beam energies such as SLC where $E \gg M$ the above equation can be approximated as

$$\overline{\mathscr{M}^2} = e^4(1 + \cos^2\theta)$$

The following script verifies that

$$\frac{1}{64E^4} \operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 - m)\gamma^\nu\right] \operatorname{Tr}\left[(\not{p}_4 - M)\gamma_\mu(\not{p}_3 + M)\gamma_\nu\right]$$
$$= 1 + \cos^2\theta + \frac{m^2 + M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta$$

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-- www.eigenmath.org/muon.txt
p = sqrt(E^2 - m^2)
rho = sqrt(E^2 - M^2)
p1 = (E,0,0,p)
p2 = (E,0,0,-p)
p3 = (E,
      rho*expsin(theta)*expcos(phi),
      rho*expsin(theta)*expsin(phi),
      rho*expcos(theta))
p4 = (E,
      -rho*expsin(theta)*expcos(phi),
      -rho*expsin(theta)*expsin(phi),
      -rho*expcos(theta))
I = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))
gmunu = ((1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma0 = ((1,0,0,0),(0,1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma1 = ((0,0,0,1),(0,0,1,0),(0,-1,0,0),(-1,0,0,0))
gamma2 = ((0,0,0,-i),(0,0,i,0),(0,i,0,0),(-i,0,0,0))
gamma3 = ((0,0,1,0),(0,0,0,-1),(-1,0,0,0),(0,1,0,0))
gamma = (gamma0,gamma1,gamma2,gamma3)
pslash1 = dot(gmunu,p1,gamma)
pslash2 = dot(gmunu,p2,gamma)
pslash3 = dot(gmunu,p3,gamma)
pslash4 = dot(gmunu,p4,gamma)
gammaT = transpose(gamma)
-- T1 is the first trace matrix
T1 = contract(dot(pslash1+m*I,gammaT,pslash2-m*I,gammaT),1,4)
-- T2 is the second trace matrix
T2 = contract(dot(pslash4-M*I,gammaT,pslash3+M*I,gammaT),1,4)
T2 = dot(gmunu,T2,gmunu)
-- A is the product of T1 and T2
A = contract(dot(T1,transpose(T2)))
-- B is the expression in theta
B = 1 + expcos(theta)^2 +
    (m^2+M^2)/E^2 * expsin(theta)^2 +
    m^2*M^2/E^4 * expcos(theta)^2
-- this difference should be zero
A/(64*E^4)-B
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The traces are 4×4 matrices. In component notation we have

$$\mathrm{Tr}\left[(\not{p}_1 + m_1)\gamma^\mu(\not{p}_2 - m_2)\gamma^\nu\right] = (\not{p}_1 + m_1)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{p}_2 - m_2)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha$$

Transpose γ to form inner products.

$$= (\not{p}_1 + m_1)^\alpha{}_\beta\gamma^{\beta\mu}{}_\rho(\not{p}_2 - m_2)^\rho{}_\sigma\gamma^{\sigma\nu}{}_\alpha$$

Convert to code. Contract sums over α.

$$= \texttt{contract(dot(}\underbrace{\texttt{pslash1+m1*I}}_{(\not{p}_1+m_1)^\alpha{}_\beta}\texttt{,}\underbrace{\texttt{gammaT}}_{\gamma^{\beta\mu}{}_\rho}\texttt{,}\underbrace{\texttt{pslash2-m2*I}}_{(\not{p}_2-m_2)^\rho{}_\sigma}\texttt{,}\underbrace{\texttt{gammaT}}_{\gamma^{\sigma\nu}{}_\alpha}\texttt{),1,4)}$$

Repeat for the second trace.

$$\operatorname{Tr}\left[(\not{p}_4 - m_4)\gamma^\mu(\not{p}_3 + m_3)\gamma^\nu\right]$$
$$= \texttt{contract(dot(pslash4-m4*I,gammaT,pslash3+m3*I,gammaT),1,4)}$$

The product of the two traces is summed over μ and ν.

$$\operatorname{Tr}[\,]\operatorname{Tr}[\,] = \operatorname{Tr}[\,]^{\mu\nu}\operatorname{Tr}[\,]_{\mu\nu}$$

Transpose the second trace to form an inner product.

$$= \operatorname{Tr}[\,]^{\mu\nu}\operatorname{Tr}[\,]_{\nu\mu}$$

Convert to code. Contract sums over μ.

$$= \texttt{contract(dot(T1,transpose(T2)))}$$