

Consider a physical system modeled by the following eigenstates.

$$\begin{aligned}
 |0\rangle &= (1\ 0\ 0\ 0)^\dagger && \text{no electrons} \\
 |1\rangle &= (0\ 1\ 0\ 0)^\dagger && \text{one electron in state } \phi_1 \\
 |2\rangle &= (0\ 0\ 1\ 0)^\dagger && \text{one electron in state } \phi_2 \\
 |3\rangle &= (0\ 0\ 0\ 1)^\dagger && \text{two electrons, one in state } \phi_1, \text{ one in state } \phi_2
 \end{aligned}$$

Let electron states ϕ_n be modeled by a one dimensional box of length L .

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Let $|\xi\rangle$ be an arbitrary normalized state vector.

$$|\xi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + c_3|3\rangle, \quad \langle\xi|\xi\rangle = 1$$

Let us determine an energy matrix \hat{E} such that the expected energy $\langle E \rangle$ in state $|\xi\rangle$ is

$$\langle E \rangle = \langle \xi | \hat{E} | \xi \rangle$$

Energy matrix \hat{E} is the sum of kinetic and potential energy matrices.

$$\hat{E} = \hat{K} + \hat{V}$$

Kinetic energy matrix \hat{K} can be computed from energy eigenvalues of the box model.

$$\hat{K} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_1 + E_2 \end{pmatrix}, \quad E_n = \frac{n^2\pi^2\hbar^2}{2m_e L^2}$$

Potential energy matrix \hat{V} has one entry due to Coulomb interaction in the two electron state.

$$\hat{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{12} \end{pmatrix}$$

Let $\psi(x, y)$ be the antisymmetrized wavefunction of the two electrons.

$$\psi(x, y) = \frac{1}{\sqrt{2}}(\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x))$$

Then

$$V_{12} = \frac{e^2}{4\pi\epsilon_0} \int_0^L \int_0^L \psi^*(x, y) \left(\frac{1}{|x-y|}\right) \psi(x, y) dx dy$$

Let us now choose $L = 10^{-9}$ meters and compute numerical values. For \hat{K} we have

$$\hat{K} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.38 \text{ eV} & 0 & 0 \\ 0 & 0 & 1.50 \text{ eV} & 0 \\ 0 & 0 & 0 & 1.88 \text{ eV} \end{pmatrix}$$

Computing V_{12} by numerical integration we have

$$\hat{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.67 \text{ eV} \end{pmatrix}$$

Hence

$$\hat{E} = \hat{K} + \hat{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.38 \text{ eV} & 0 & 0 \\ 0 & 0 & 1.50 \text{ eV} & 0 \\ 0 & 0 & 0 & 6.55 \text{ eV} \end{pmatrix}$$