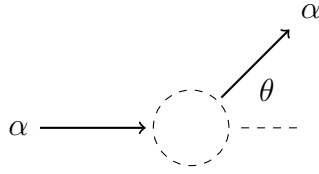
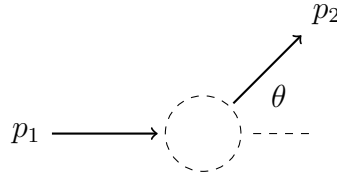


RUTHERFORD SCATTERING

Alpha particles are scattered by atomic nuclei due to the Coulomb force.



This is the same diagram with momentum labels.



These are the momentum vectors for p_1 aligned with the z axis and with $p = \sqrt{E^2 - m^2}$.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix}$$

The spin averaged and scaled probability density $f(\theta, \phi) = \langle \mathcal{M}^* \mathcal{M} \rangle$ is computed from the momentum vectors as follows.

$$\begin{aligned} f(\theta, \phi) &= \frac{C}{4} \sum_{s_1=1}^2 \sum_{s_2=1}^2 |\bar{u}_2 \gamma^0 u_1|^2 \\ &= \frac{C}{4} \text{Tr} [(\not{p}_1 + m) \gamma^0 (\not{p}_2 + m) \gamma^0] \\ &= C (E^2 + m^2 + p^2 \cos \theta) \end{aligned}$$

where

$$C = \frac{Z^2 e^4}{8p^4 \sin^4 \theta / 2}$$

The following script verifies that

$$\frac{1}{4} \text{Tr} [(\not{p}_1 + m) \gamma^0 (\not{p}_2 + m) \gamma^0] = E^2 + m^2 + p^2 \cos \theta$$

```
-- www.eigenmath.org/coulomb.txt
p = sqrt(E^2 - m^2)
p1 = (E,0,0,p)
p2 = (E,
      p*expsin(theta)*expcos(phi),
      p*expsin(theta)*expsin(phi),
      p*expcos(theta))
I = ((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1))
```

```

gmunu = ((1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma0 = ((1,0,0,0),(0,1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma1 = ((0,0,0,1),(0,0,1,0),(0,-1,0,0),(-1,0,0,0))
gamma2 = ((0,0,0,-i),(0,0,i,0),(0,i,0,0),(-i,0,0,0))
gamma3 = ((0,0,1,0),(0,0,0,-1),(-1,0,0,0),(0,1,0,0))
gamma = (gamma0,gamma1,gamma2,gamma3)
pslash1 = dot(gmunu,p1,gamma)
pslash2 = dot(gmunu,p2,gamma)
A = contract(dot(pslash1+m*I,gamma0,pslash2+m*I,gamma0))
B = E^2 + m^2 + p^2*expcos(theta)
-- this difference should be zero
A/4-B

```

The following script shows that

$$\frac{E^2 + m^2 + p^2 \cos \theta}{8p^4} = \frac{1 - \beta^2 \sin^2 \theta/2}{4p^2 \beta^2}$$

where $p = \sqrt{E^2 - m^2}$ and $\beta = p/E$. The exponential forms of sine and cosine are used so that trigonometric identities are not required.

```

p = sqrt(E^2 - m^2)
beta = p/E
A = (E^2 + m^2 + p^2 * expcos(theta)) / (8*p^4)
B = (1 - beta^2 * expsin(theta/2)^2) / (4*p^2*beta^2)
A-B

```