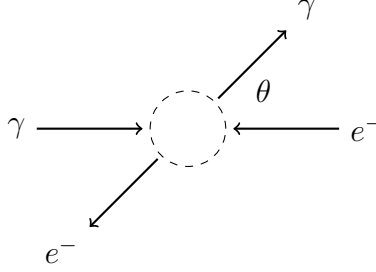
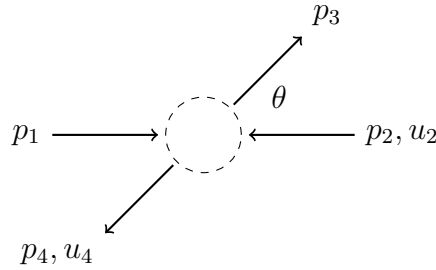


COMPTON SCATTERING

Compton scattering occurs when a gamma ray hits an electron.



Here is the same diagram with momentum and spinor labels.



In center of mass coordinates the momentum vectors are

$$p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix} \quad p_3 = \begin{pmatrix} \omega \\ \omega \sin \theta \cos \phi \\ \omega \sin \theta \sin \phi \\ \omega \cos \theta \end{pmatrix} \quad p_4 = \begin{pmatrix} E \\ -\omega \sin \theta \cos \phi \\ -\omega \sin \theta \sin \phi \\ -\omega \cos \theta \end{pmatrix}$$

where $E = \sqrt{\omega^2 + m^2}$. The spinors are

$$u_{21} = \begin{pmatrix} E + m \\ 0 \\ -\omega \\ 0 \end{pmatrix} \quad u_{41} = \begin{pmatrix} E + m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}$$

$$u_{22} = \begin{pmatrix} 0 \\ E + m \\ 0 \\ \omega \end{pmatrix} \quad u_{42} = \begin{pmatrix} 0 \\ E + m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E + m)^2$ will be used where needed.

This is the probability density for Compton scattering. The formula is from Feynman diagrams. Symbol s_j selects the spin of spinor j . Symbol e is electron charge. Symbols s and u are Mandelstam variables $s = (p_1 + p_2)^2$ and $u = (p_1 - p_4)^2$. Symbol $q_1 = p_1 + p_2$ and $q_2 = p_2 - p_3$.

$$|\mathcal{M}(s_2, s_4)|^2 = \frac{e^4}{N} \left| -\frac{\bar{u}_4 \gamma^\mu (\not{q}_1 + m) \gamma^\nu u_2}{s - m^2} - \frac{\bar{u}_4 \gamma^\nu (\not{q}_2 + m) \gamma^\mu u_2}{u - m^2} \right|^2$$

Let

$$a_1 = \bar{u}_4 \gamma^\mu (\not{p}_1 + m) \gamma^\nu u_2 \quad a_2 = \bar{u}_4 \gamma^\nu (\not{p}_2 + m) \gamma^\mu u_2$$

Then

$$\begin{aligned} |\mathcal{M}(s_2, s_4)|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2} \right|^2 \\ &= \frac{e^4}{N} \left(-\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2} \right) \left(-\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2} \right)^* \\ &= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{(s-m^2)^2} + \frac{a_1 a_2^*}{(s-m^2)(u-m^2)} + \frac{a_1^* a_2}{(s-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2} \right) \end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarizations is already accomplished by contraction of aa^* over μ and ν .

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{s_2=1}^2 \sum_{s_4=1}^2 |\mathcal{M}(s_2, s_4)|^2 \\ &= \frac{e^4}{4} \sum_{s_2=1}^2 \sum_{s_4=1}^2 \frac{1}{N} \left(\frac{a_1 a_1^*}{(s-m^2)^2} + \frac{a_1 a_2^*}{(s-m^2)(u-m^2)} + \frac{a_1^* a_2}{(s-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2} \right) \end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned} f_{11} &= \frac{1}{N} \sum_{\text{spins}} a_1 a_1^* = \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{p}_1 + m) \gamma_\mu \right) \\ f_{12} &= \frac{1}{N} \sum_{\text{spins}} a_1 a_2^* = \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right) \\ f_{22} &= \frac{1}{N} \sum_{\text{spins}} a_2 a_2^* = \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{p}_2 + m) \gamma_\mu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{(s-m^2)^2} + \frac{f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{12}^*}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2} \right) \quad (1)$$

Run “compton-scattering-1.txt” to verify the Casimir trick for Compton scattering.

These formulas compute probability densities from dot products. Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$.

$$\begin{aligned} f_{11} &= -16(p_1 \cdot p_1)(p_2 \cdot p_4) + 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_2) + 16(p_2 \cdot p_2)(p_2 \cdot p_4) \\ &\quad + 64m^2(p_1 \cdot p_1) + 64m^2(p_1 \cdot p_2) - 64m^2(p_1 \cdot p_4) - 48m^2(p_2 \cdot p_4) + 64m^4 \\ f_{12} &= -32(p_1 \cdot p_2)(p_2 \cdot p_4) + 32(p_1 \cdot p_3)(p_2 \cdot p_4) - 32(p_2 \cdot p_2)(p_2 \cdot p_4) + 32(p_2 \cdot p_3)(p_2 \cdot p_4) \\ &\quad + 32m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) \\ &\quad + 48m^2(p_2 \cdot p_2) - 32m^2(p_2 \cdot p_3) + 48m^2(p_2 \cdot p_4) - 16m^2(p_3 \cdot p_4) - 32m^4 \\ f_{22} &= 16(p_2 \cdot p_2)(p_2 \cdot p_4) - 32(p_2 \cdot p_2)(p_3 \cdot p_4) + 32(p_2 \cdot p_3)(p_3 \cdot p_4) - 16(p_2 \cdot p_4)(p_3 \cdot p_3) \\ &\quad - 64m^2(p_2 \cdot p_3) - 48m^2(p_2 \cdot p_4) + 64m^2(p_3 \cdot p_3) + 64m^2(p_3 \cdot p_4) + 64m^4 \end{aligned}$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ the formulas are

$$\begin{aligned} f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\ f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\ f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4 \end{aligned} \tag{2}$$

Run “compton-scattering-2.txt” to verify.

Compton scattering experiments are done in a “lab” frame where the electron is at rest. The following Lorentz boost Λ transforms momentum vectors from the center of mass frame to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}, \quad \Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\begin{aligned} \omega_L &= \Lambda p_1 \cdot (1, 0, 0, 0) = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot (1, 0, 0, 0) = \frac{\omega^2 \cos \theta}{m} + \frac{\omega E}{m} \end{aligned}$$

It follows that

$$\begin{aligned} s &= (p_1 + p_2)^2 = 2m\omega_L + m^2 \\ t &= (p_1 - p_3)^2 = 2m(\omega'_L - \omega_L) \\ u &= (p_1 - p_4)^2 = -2m\omega'_L + m^2 \end{aligned}$$

Compute $\langle |\mathcal{M}|^2 \rangle$ using equations (1) and (2) and the above s , t , and u that involve ω_L and ω'_L .

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

From the Compton formula

$$\frac{1}{\omega'_L} - \frac{1}{\omega_L} = \frac{1 - \cos \theta_L}{m}$$

we have

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right)$$

Run “compton-scattering-3.txt” to verify.

Now that we have $\langle |\mathcal{M}|^2 \rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω , ω' , and θ will be lab frame variables.

The differential cross section for Compton scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2 \theta - 1 \right)$$

From the Compton equation we have

$$\omega' = \frac{m\omega}{m + \omega(1 - \cos \theta)}$$

Use the Compton equation to eliminate ω' in $d\sigma$.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} \left(\frac{m}{m + \omega(1 - \cos \theta)} \right)^2 \left(\frac{m + \omega(1 - \cos \theta)}{m} + \frac{m}{m + \omega(1 - \cos \theta)} + \cos^2 \theta - 1 \right)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function.

Let

$$I(\xi) = 2\pi \int_0^\xi \frac{d\sigma}{d\Omega} \sin \theta d\theta, \quad 0 \leq \xi \leq \pi$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)}{I(\pi)}, \quad 0 \leq \theta \leq \pi$$

Hence

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

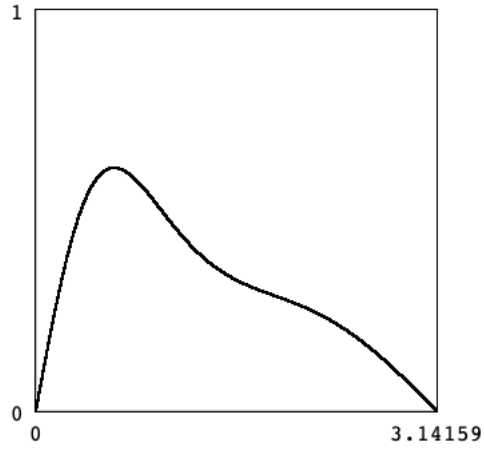
The probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{2\pi}{I(\pi)} \left(\frac{d\sigma}{d\Omega} \right) \sin \theta, \quad 0 \leq \theta \leq \pi$$

The following angular distribution is for $\omega = m$.

θ_1	θ_2	$P(\theta_1 \leq \theta \leq \theta_2)$
0°	45°	0.35
45°	90°	0.34
90°	135°	0.22
135°	180°	0.09

Run “compton-scattering-4.txt” to plot $f(\theta)$.



Plot of $f(\theta)$ for $\omega = m$.

When ω is much smaller than the electron mass m we have

$$\frac{m}{m + \omega(1 - \cos \theta)} \approx 1$$

Hence for $\omega \ll m$ the differential cross section is approximately

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}(1 + \cos^2 \theta)$$

which is the formula for Thomson scattering.

Here are some notes on how the scripts work.

To convert a_1 and a_2 to Eigenmath code, it is instructive to write a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha} \gamma^{\mu\alpha}{}_{\beta} (\not{d}_1 + m)^{\beta}{}_{\rho} \gamma^{\nu\rho}{}_{\sigma} u_2^{\sigma} \quad a_2^{\nu\mu} = \bar{u}_{4\alpha} \gamma^{\nu\alpha}{}_{\beta} (\not{d}_2 + m)^{\beta}{}_{\rho} \gamma^{\mu\rho}{}_{\sigma} u_2^{\sigma}$$

Transpose γ tensors to form inner products over α and ρ .

$$a_1^{\mu\nu} = \bar{u}_{4\alpha} \gamma^{\alpha\mu}{}_{\beta} (\not{d}_1 + m)^{\beta}{}_{\rho} \gamma^{\rho\nu}{}_{\sigma} u_2^{\sigma} \quad a_2^{\nu\mu} = \bar{u}_{4\alpha} \gamma^{\alpha\nu}{}_{\beta} (\not{d}_2 + m)^{\beta}{}_{\rho} \gamma^{\rho\mu}{}_{\sigma} u_2^{\sigma}$$

Convert transposed γ to Eigenmath code.

$$\gamma^{\alpha\mu}{}_{\beta} \rightarrow \text{gammaT} = \text{transpose}(\text{gamma})$$

Then to compute a_1 we have

$$a_1 = \bar{u}_{4\alpha} \gamma^{\alpha\mu}{}_{\beta} (\not{d}_1 + m)^{\beta}{}_{\rho} \gamma^{\rho\nu}{}_{\sigma} u_2^{\sigma} \\ \rightarrow \text{a1} = \text{dot}(\text{u4bar}[\text{s4}], \text{gammaT}, \text{qslash1} + \text{m I}, \text{gammaT}, \text{u2}[\text{s2}])$$

where s_2 and s_4 are spin indices. Similarly for a_2 we have

$$a_2 = \bar{u}_{4\alpha} \gamma^{\alpha\nu}{}_{\beta} (\not{d}_2 + m)^{\beta}{}_{\rho} \gamma^{\rho\mu}{}_{\sigma} u_2^{\sigma} \\ \rightarrow \text{a2} = \text{dot}(\text{u4bar}[\text{s4}], \text{gammaT}, \text{qslash2} + \text{m I}, \text{gammaT}, \text{u2}[\text{s2}])$$

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν .

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ .

$$a_1 a_1^* \rightarrow \text{a11} = \text{contract}(\text{dot}(\text{a1}, \text{gmunu}, \text{transpose}(\text{conj}(\text{a1}))), \text{gmunu})$$

Similarly for $a_2 a_2^*$ we have

$$a_2 a_2^* \rightarrow \text{a22} = \text{contract}(\text{dot}(\text{a2}, \text{gmunu}, \text{transpose}(\text{conj}(\text{a2}))), \text{gmunu})$$

The product $a_1 a_2^*$ does not require a transpose because $a_1 a_2^* = a_1^{\mu\nu} a_2^{*\nu\mu}$.

$$a_1 a_2^* \rightarrow \text{a12} = \text{contract}(\text{dot}(\text{a1}, \text{gmunu}, \text{conj}(\text{a2})), \text{gmunu})$$

In component notation, a trace operator becomes a sum over an index, in this case α .

$$\begin{aligned} f_{11} &= \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ &= (\not{p}_2 + m)^\alpha_\beta \gamma^{\mu\beta}_\rho (\not{q}_1 + m)^\rho_\sigma \gamma^{\nu\sigma}_\tau (\not{p}_4 + m)^\tau_\delta \gamma^{\delta}_\eta (\not{q}_1 + m)^\eta_\xi \gamma^\xi_\alpha \end{aligned}$$

As before, transpose γ tensors to form inner products.

$$f_{11} = (\not{p}_2 + m)^\alpha_\beta \gamma^{\beta\mu}_\rho (\not{q}_1 + m)^\rho_\sigma \gamma^{\sigma\nu}_\tau (\not{p}_4 + m)^\tau_\delta \gamma^{\delta}_\nu (\not{q}_1 + m)^\eta_\xi \gamma^\xi_{\mu\alpha}$$

To convert to Eigenmath code, use an intermediate variable for the inner product.

$$T^{\alpha\mu\nu}_{\nu\mu\alpha} \quad \rightarrow \quad \mathbf{T} = \text{dot}(\mathbf{P2}, \mathbf{gammaT}, \mathbf{Q1}, \mathbf{gammaT}, \mathbf{P4}, \mathbf{gammaL}, \mathbf{Q1}, \mathbf{gammaL})$$

Now sum over the indices of T . The innermost contract sums over ν then the next contract sums over μ . Finally the outermost contract sums over α .

$$f_{11} \quad \rightarrow \quad \mathbf{f11} = \text{contract}(\text{contract}(\text{contract}(\mathbf{T}, 3, 4), 2, 3))$$

Follow suit for f_{22} . For f_{12} the order of the rightmost μ and ν is reversed.

$$f_{12} = \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right)$$

The resulting inner product is $T^{\alpha\mu\nu}_{\mu\nu\alpha}$ so the contraction is different.

$$f_{12} \quad \rightarrow \quad \mathbf{f12} = \text{contract}(\text{contract}(\text{contract}(\mathbf{T}, 3, 5), 2, 3))$$

The innermost contract sums over ν followed by sum over μ then sum over α .