Bhabha scattering is the interaction between positrons and electrons.

\[
\begin{align*}
& e^+ & \theta & e^- \\
& e^- & \theta & e^+
\end{align*}
\]

Here is the same diagram with momentum and spinor labels.

\[
\begin{align*}
&p_1, v_1 & p_2, u_2 & p_3, v_3 & p_4, u_4 \\
& p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} & p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} & p_3 = \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix} & p_4 = \begin{pmatrix} E \\ -p \sin \theta \cos \phi \\ -p \sin \theta \sin \phi \\ -p \cos \theta \end{pmatrix}
\end{align*}
\]

In a typical collider experiment the momentum vectors are

where \( p = \sqrt{E^2 - m^2} \). The spinors are

\[
\begin{align*}
& v_{11} = \begin{pmatrix} p \\ 0 \\ E + m \\ 0 \end{pmatrix} & u_{21} = \begin{pmatrix} E + m \\ 0 \\ -p \\ 0 \end{pmatrix} & v_{31} = \begin{pmatrix} p_3^x \\ p_3^x + ip_3^y \\ E + m \\ 0 \end{pmatrix} & u_{41} = \begin{pmatrix} E + m \\ 0 \\ p_4^x \\ p_4^x + ip_4^y \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
& v_{12} = \begin{pmatrix} 0 \\ -p \\ 0 \\ E + m \end{pmatrix} & u_{22} = \begin{pmatrix} E + m \\ 0 \\ 0 \\ p \end{pmatrix} & v_{32} = \begin{pmatrix} p_3^x - ip_3^y \\ -p_3^x \\ 0 \\ E + m \end{pmatrix} & u_{42} = \begin{pmatrix} 0 \\ E + m \\ 0 \\ p_4^x - ip_4^y \end{pmatrix}
\end{align*}
\]

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant \( N = (E + m)^4 \) will be used where needed.
This is the probability density for Bhabha scattering. The formula is from Feynman diagrams.

\[ |M(s_1, s_2, s_3, s_4)|^2 = \frac{e^4}{N} \left| -\frac{1}{t} (\bar{v}_1 \gamma^\mu v_3)(\bar{u}_4 \gamma_\mu u_2) + \frac{1}{s} (\bar{v}_1 \gamma^\nu u_2)(\bar{u}_4 \gamma_\nu v_3) \right|^2 \]

Symbol \( s_j \) selects the spin (up or down) of spinor \( j \). Symbol \( e \) is electron charge. Symbols \( s \) and \( t \) are Mandelstam variables \( s = (p_1 + p_2)^2 \) and \( t = (p_1 - p_3)^2 \).

Let

\[ a_1 = (\bar{v}_1 \gamma^\mu v_3)(\bar{u}_4 \gamma_\mu u_2) \quad a_2 = (\bar{v}_1 \gamma^\nu u_2)(\bar{u}_4 \gamma_\nu v_3) \]

Then

\[ |M(s_1, s_2, s_3, s_4)|^2 = \frac{e^4}{N} \left| -\frac{a_1}{t} + \frac{a_2}{s} \right|^2 = \frac{e^4}{N} \left( -\frac{a_1}{t} + \frac{a_2}{s} \right) \left( -\frac{a_1^*}{t} + \frac{a_2^*}{s} \right) = \frac{e^4}{N} \left( \frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{st} - \frac{a_1^* a_2}{st} + \frac{a_2 a_2^*}{s^2} \right) \]

The expected probability density \( \langle |M|^2 \rangle \) is computed by summing \( |M|^2 \) over all spin states and dividing by the number of inbound states. There are four inbound states.

\[ \langle |M|^2 \rangle = \frac{1}{4} \sum_{s_1 = 1}^{2} \sum_{s_2 = 1}^{2} \sum_{s_3 = 1}^{2} \sum_{s_4 = 1}^{2} |M(s_1, s_2, s_3, s_4)|^2 \]

\[ = \frac{e^4}{4} \sum_{s_1 = 1}^{2} \sum_{s_2 = 1}^{2} \sum_{s_3 = 1}^{2} \sum_{s_4 = 1}^{2} \frac{1}{N} \left( \frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{st} - \frac{a_1^* a_2}{st} + \frac{a_2 a_2^*}{s^2} \right) \]

Use the Casimir trick to replace sums over spins with matrix products.

\[ f_{11} = \frac{1}{N} \sum_{\text{spins}} a_1 a_1^* = \text{Tr} \left( (\gamma_1 - m)\gamma^\mu (\gamma_3 - m)\gamma^\nu \right) \text{Tr} \left( (\gamma_4 + m)\gamma_\mu (\gamma_2 + m)\gamma_\nu \right) \]

\[ f_{12} = \frac{1}{N} \sum_{\text{spins}} a_1 a_2^* = \text{Tr} \left( (\gamma_1 - m)\gamma^\mu (\gamma_2 + m)\gamma^\nu (\gamma_4 + m)\gamma_\mu (\gamma_3 - m)\gamma_\nu \right) \]

\[ f_{22} = \frac{1}{N} \sum_{\text{spins}} a_2 a_2^* = \text{Tr} \left( (\gamma_1 - m)\gamma^\mu (\gamma_2 + m)\gamma^\nu \right) \text{Tr} \left( (\gamma_4 + m)\gamma_\mu (\gamma_3 - m)\gamma_\nu \right) \]

Hence

\[ \langle |M|^2 \rangle = \frac{e^4}{4} \left( \frac{f_{11}}{t^2} - \frac{f_{12}}{st} - \frac{f_{12}^*}{st} + \frac{f_{22}}{s^2} \right) \]

Run “bhabha-scattering-1.txt” to verify the Casimir trick.
These formulas compute probability densities from dot products.

\[ f_{11} = 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) - 32m^2(p_1 \cdot p_3) - 32m^2(p_2 \cdot p_4) + 64m^4 \]
\[ f_{12} = -32(p_1 \cdot p_4)(p_2 \cdot p_3) - 16m^2(p_1 \cdot p_2) + 16m^2(p_1 \cdot p_3) - 16m^2(p_1 \cdot p_4) \]
\[ - 16m^2(p_2 \cdot p_3) + 16m^2(p_2 \cdot p_4) - 16m^2(p_3 \cdot p_4) - 32m^4 \]
\[ f_{22} = 32(p_1 \cdot p_3)(p_2 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) + 32m^2(p_1 \cdot p_2) + 32m^2(p_3 \cdot p_4) + 64m^4 \]

In Mandelstam variables \( s = (p_1 + p_2)^2, \ t = (p_1 - p_3)^2, \ u = (p_1 - p_4)^2 \) the formulas are

\[ f_{11} = 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \]
\[ f_{12} = -8u^2 + 64um^2 - 96m^4 \]
\[ f_{22} = 8t^2 + 8u^2 - 64tm^2 - 64um^2 + 192m^4 \]

When \( E \gg m \) a useful approximation is to set \( m = 0 \) and obtain

\[ f_{11} = 8s^2 + 8u^2 \]
\[ f_{12} = -8u^2 \]
\[ f_{22} = 8t^2 + 8u^2 \]

For \( m = 0 \) the Mandelstam variables are

\[ s = 4E^2 \]
\[ t = -2E^2(1 - \cos \theta) = -4E^2 \sin^2(\theta/2) \]
\[ u = -2E^2(1 + \cos \theta) = -4E^2 \cos^2(\theta/2) \]

The corresponding expected probability density is

\[ \langle |M|^2 \rangle = \frac{e^4}{4} \left( \frac{8s^2 + 8u^2}{t^2} + \frac{16u^2}{st} + \frac{8t^2 + 8u^2}{s^2} \right) \]
\[ = 2e^4 \left( \frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2 + u^2}{s^2} \right) \]
\[ = 2e^4 \left( \frac{1 + \cos^2(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1 + \cos^2 \theta}{2} \right) \]

Run “bhabha-scattering-2.txt” to verify.
This is the differential cross section for Bhabha scattering.

\[
\frac{d\sigma}{d\Omega} = \frac{\langle |M|^2 \rangle}{64\pi^2s} = \frac{\alpha^2}{8E^2} \left( \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1 + \cos^2\theta}{2} \right)
\]

We can integrate \(d\sigma\) to obtain a cumulative distribution function.

Let

\[I(\xi) = 2\pi \int_{\alpha}^{\xi} \frac{d\sigma}{d\Omega} \sin\theta d\theta, \quad \alpha \leq \xi \leq \pi\]

for some \(\alpha > 0\). The support interval is restricted because \(d\sigma\) is undefined for \(\theta = 0\).

The cumulative distribution function is

\[F(\theta) = \frac{I(\theta)}{I(\pi)}, \quad \alpha \leq \theta \leq \pi\]

Hence

\[P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)\]

The probability density is

\[f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{\sin\theta}{I(\pi)} \left( \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1 + \cos^2\theta}{2} \right), \quad \alpha \leq \theta \leq \pi\]

Run “bhabha-scattering-3.txt” to draw \(f(\theta)\) for \(\alpha = \pi/180\).

Here is a probability distribution for 45° bins with \(\alpha = 45^\circ\).

<table>
<thead>
<tr>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>(P(\theta_1 \leq \theta \leq \theta_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>45°</td>
<td>–</td>
</tr>
<tr>
<td>45°</td>
<td>90°</td>
<td>0.83</td>
</tr>
<tr>
<td>90°</td>
<td>135°</td>
<td>0.13</td>
</tr>
<tr>
<td>135°</td>
<td>180°</td>
<td>0.04</td>
</tr>
</tbody>
</table>
The following Bhabha scattering data is adapted from SLAC-PUB-1501.

<table>
<thead>
<tr>
<th>Bin</th>
<th>cos θ (interval)</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Smallest θ)</td>
<td>1</td>
<td>0.6, 0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5, 0.4</td>
<td>2841</td>
</tr>
<tr>
<td>3</td>
<td>0.4, 0.3</td>
<td>2045</td>
</tr>
<tr>
<td>4</td>
<td>0.3, 0.2</td>
<td>1420</td>
</tr>
<tr>
<td>5</td>
<td>0.2, 0.1</td>
<td>1136</td>
</tr>
<tr>
<td>6</td>
<td>0.1, 0.0</td>
<td>852</td>
</tr>
<tr>
<td>7</td>
<td>0.0, −0.1</td>
<td>656</td>
</tr>
<tr>
<td>8</td>
<td>−0.1, −0.2</td>
<td>625</td>
</tr>
<tr>
<td>9</td>
<td>−0.2, −0.3</td>
<td>511</td>
</tr>
<tr>
<td>10</td>
<td>−0.3, −0.4</td>
<td>455</td>
</tr>
<tr>
<td>11</td>
<td>−0.4, −0.5</td>
<td>402</td>
</tr>
<tr>
<td>12</td>
<td>−0.5, −0.6</td>
<td>398</td>
</tr>
<tr>
<td>(Largest θ)</td>
<td>12</td>
<td>−0.5, −0.6</td>
</tr>
</tbody>
</table>

“Count” is the number of Bhabha scattering events observed per bin. Let us see if the density function \(|\langle M \rangle^2\rangle\) explains the distribution of counts in the table. Start by integrating \(|\langle M \rangle^2\rangle\) over all the bins to obtain a normalization constant.

\[
\int_{\text{bins}} |\langle M \rangle^2\rangle\, d\Omega = \int_0^{2\pi} \int_{\arccos 0.6}^{\arccos 0.6} |\langle M \rangle^2\rangle\sin \theta\, d\theta\, d\phi = 2\pi \times 9.3817 \times 2e^4
\]

Let

\[
\frac{|\langle M \rangle^2\rangle}{2\pi \times 9.3817 \times 2e^4} = \frac{1}{2\pi \times 9.3817} \left( \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} - 2\frac{\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1 + \cos^2 \theta}{2} \right)
\]

The probability of a scattering event occurring in an interval \(\theta_1\) to \(\theta_2\) is obtained by integrating \(f(\theta)\) over that interval.

\[
P(\theta_1 < \theta < \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} f(\theta)\sin \theta\, d\theta\, d\phi = 2\pi \int_{\theta_1}^{\theta_2} f(\theta)\sin \theta\, d\theta
\]

The total number of counts in the table is 15773. To obtain a predicted distribution, multiply 15773 times the probability for each bin. For example, for the first bin we have

\[
P(\arccos 0.6 < \theta < \arccos 0.5) \times 15773 = 4598
\]

Repeat for all bins to obtain the following predicted distribution.
The coefficient of determination $R^2$ measures how well predicted values fit the real data. Let $y$ be observed counts per bin and let $\hat{y}$ be predicted counts per bin. Then

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.997$$

The result indicates that the model $\langle |\mathcal{M}|^2 \rangle$ explains 99.7% of the variance in the data.

Run “bhabha-scattering-4.txt” to verify.
The following table shows DESY-PETRA Bhabha scattering data obtained from HEP Data.\(^1\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.73</td>
<td>0.10115</td>
</tr>
<tr>
<td>-0.6495</td>
<td>0.12235</td>
</tr>
<tr>
<td>-0.5495</td>
<td>0.11258</td>
</tr>
<tr>
<td>-0.4494</td>
<td>0.09968</td>
</tr>
<tr>
<td>-0.3493</td>
<td>0.14749</td>
</tr>
<tr>
<td>-0.2491</td>
<td>0.14017</td>
</tr>
<tr>
<td>-0.149</td>
<td>0.1819</td>
</tr>
<tr>
<td>-0.0488</td>
<td>0.22964</td>
</tr>
<tr>
<td>0.0514</td>
<td>0.25312</td>
</tr>
<tr>
<td>0.1516</td>
<td>0.30998</td>
</tr>
<tr>
<td>0.252</td>
<td>0.40898</td>
</tr>
<tr>
<td>0.3524</td>
<td>0.62695</td>
</tr>
<tr>
<td>0.4529</td>
<td>0.91803</td>
</tr>
<tr>
<td>0.5537</td>
<td>1.51743</td>
</tr>
<tr>
<td>0.6548</td>
<td>2.56714</td>
</tr>
<tr>
<td>0.7323</td>
<td>4.30279</td>
</tr>
</tbody>
</table>

Data \(x\) and \(y\) have the following relationship with the cross section model.

\[
x = \cos \theta \quad y = \frac{d\sigma}{d\Omega}
\]

The differential cross section for Bhabha scattering is

\[
\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{\alpha^2}{2s} \left( \frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2 + u^2}{s^2} \right)
\]

The predicted cross section \(\hat{y}\) is computed from data \(x\) and beam energy \(E\) as

\[
\hat{y} = \frac{\alpha^2}{2s} \left( \frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2 + u^2}{s^2} \right) \times (hc)^2 \times 10^{37}
\]

where

\[
s = 4E^2 \\
t = -2E^2(1 - x) \\
u = -2E^2(1 + x)
\]

Factor \((hc)^2\) converts the result to SI and factor \(10^{37}\) converts square meters to nanobarns.

The following table shows \(\hat{y}\) for \(E = 7.0\text{ GeV}\).

\(^1\)www.hepdata.net/record/ins191231 (Table 3, 14.0 GeV)
The coefficient of determination $R^2$ measures how well predicted values fit the real data.

\[
R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.995
\]

The result indicates that the model $d\sigma$ explains 99.5% of the variance in the data.

Run “bhabha-scattering-5.txt” to verify.
Here are a few notes about how the scripts work. In component notation the trace operators of the Casimir trick become sums over the repeated index $\alpha$.

\[
\begin{align*}
 f_{11} &= (\psi_1 - m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_3 - m)^{\rho \sigma} \gamma^\sigma_{\alpha} \\
 f_{12} &= (\psi_1 - m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_2 + m)^{\rho \sigma} \gamma^\sigma (\psi_4 + m)^{\tau} \delta_{\mu}^\tau \gamma^\delta_{\eta} (\psi_3 - m)^{\eta \xi} \gamma^{\xi \alpha} \\
 f_{22} &= (\psi_1 - m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_2 + m)^{\rho \sigma} \gamma^\sigma_{\alpha} \\
 &\quad \times (\psi_4 + m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_3 - m)^{\rho \sigma} \gamma^\sigma_{\alpha}
\end{align*}
\]

To convert the above formulas to Eigenmath code, the $\gamma$ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply $\gamma^\mu$ by the metric tensor to lower the index.

\[
\begin{align*}
\gamma^{\beta \mu}_{\rho} &\rightarrow \gamma^{\beta \mu}_{\rho} \rightarrow \text{gammaT} = \text{transpose}(\gamma) \\
\gamma^{\beta \mu}_{\rho} &\rightarrow \gamma^{\beta \mu}_{\rho} \rightarrow \text{gammaL} = \text{transpose}(\text{dot}(\text{gmunu}, \gamma))
\end{align*}
\]

Define the following $4 \times 4$ matrices.

\[
\begin{align*}
(\psi_1 - m) &\rightarrow X_1 = \text{pslash1} - m \text{ I} \\
(\psi_2 + m) &\rightarrow X_2 = \text{pslash2} + m \text{ I} \\
(\psi_3 - m) &\rightarrow X_3 = \text{pslash3} - m \text{ I} \\
(\psi_4 + m) &\rightarrow X_4 = \text{pslash4} + m \text{ I}
\end{align*}
\]

Then for $f_{11}$ we have the following Eigenmath code. The contract function sums over $\alpha$.

\[
\begin{align*}
(\psi_1 - m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_3 - m)^{\rho \sigma} \gamma^\sigma_{\alpha} &\rightarrow T_1 = \text{contract}(\text{dot}(X_1, \text{gammaT}, X_3, \text{gammaT}), 1, 4) \\
(\psi_4 + m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_2 + m)^{\rho \sigma} \gamma^\sigma_{\alpha} &\rightarrow T_2 = \text{contract}(\text{dot}(X_4, \text{gammaL}, X_2, \text{gammaL}), 1, 4)
\end{align*}
\]

Next, multiply then sum over repeated indices. The dot function sums over $\nu$ then the contract function sums over $\mu$. The transpose makes the $\nu$ indices adjacent as required by the dot function.

\[
\begin{align*}
f_{11} = \text{Tr}(\cdots \gamma^{\mu} \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_{\mu} \cdots \gamma_{\nu}) &\rightarrow f_{11} = \text{contract}(\text{dot}(T_1, \text{transpose}(T_2)))
\end{align*}
\]

Follow suit for $f_{22}$.

\[
\begin{align*}
(\psi_1 - m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_2 + m)^{\rho \sigma} \gamma^\sigma_{\alpha} &\rightarrow T_1 = \text{contract}(\text{dot}(X_1, \text{gammaT}, X_2, \text{gammaT}), 1, 4) \\
(\psi_4 + m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_3 - m)^{\rho \sigma} \gamma^\sigma_{\alpha} &\rightarrow T_2 = \text{contract}(\text{dot}(X_4, \text{gammaL}, X_3, \text{gammaL}), 1, 4)
\end{align*}
\]

Hence

\[
\begin{align*}
f_{22} = \text{Tr}(\cdots \gamma^{\mu} \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_{\mu} \cdots \gamma_{\nu}) &\rightarrow f_{22} = \text{contract}(\text{dot}(T_1, \text{transpose}(T_2)))
\end{align*}
\]

The calculation of $f_{12}$ begins with

\[
(\psi_1 - m)^{\alpha \beta} \gamma^{\mu \rho} (\psi_2 + m)^{\rho \sigma} \gamma^\sigma (\psi_4 + m)^{\tau} \delta_{\mu}^\tau \gamma^\delta_{\eta} (\psi_3 - m)^{\eta \xi} \gamma^{\xi \alpha}
\]

\[
\rightarrow T = \text{contract}(\text{dot}(X_1, \text{gammaT}, X_2, \text{gammaT}, X_4, \text{gammaL}, X_3, \text{gammaL}), 1, 6)
\]

Then sum over repeated indices $\mu$ and $\nu$.

\[
\begin{align*}
f_{12} = \text{Tr}(\cdots \gamma^{\mu} \cdots \gamma^\nu \cdots) &\rightarrow f_{12} = \text{contract}(\text{contract}(T, 1, 3))
\end{align*}
\]

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