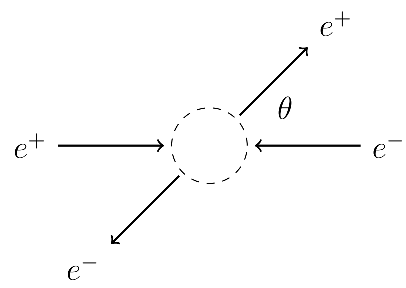
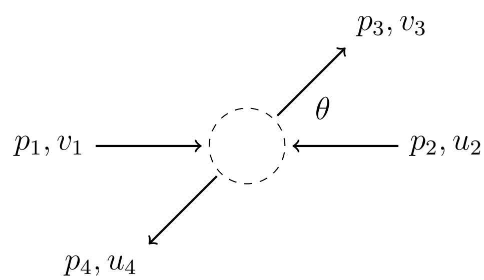
Bhabha scattering is the interaction between positrons and electrons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}$$

where $p = \sqrt{E^2 - m^2}$. The spinors are

$$v_{11} = \begin{pmatrix} p \\ 0 \\ E+m \\ 0 \end{pmatrix} \quad u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \quad v_{31} = \begin{pmatrix} p_3^z \\ p_3^x + ip_3^y \\ E+m \\ 0 \end{pmatrix} \quad u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}$$

$$v_{12} = \begin{pmatrix} 0 \\ -p \\ 0 \\ E+m \end{pmatrix} \quad u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix} \quad v_{32} = \begin{pmatrix} p_3^x - ip_3^y \\ -p_3^z \\ 0 \\ E+m \end{pmatrix} \quad u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used where needed.

This is the probability density for Bhabha scattering. The formula is from Feynman diagrams.

$$|\mathcal{M}(s_1,s_2,s_3,s_4)|^2 = \frac{e^4}{N}\left|-\frac{1}{t}(\bar v_1\gamma^\mu v_3)(\bar u_4\gamma_\mu u_2) + \frac{1}{s}(\bar v_1\gamma^\nu u_2)(\bar u_4\gamma_\nu v_3)\right|^2$$

Symbol s_j selects the spin (up or down) of spinor j. Symbol e is electron charge. Symbols s and t are Mandelstam variables $s=(p_1+p_2)^2$ and $t=(p_1-p_3)^2$.

Let

$$a_1 = (\bar v_1\gamma^\mu v_3)(\bar u_4\gamma_\mu u_2) \qquad a_2 = (\bar v_1\gamma^\nu u_2)(\bar u_4\gamma_\nu v_3)$$

Then

$$\begin{aligned}
|\mathcal{M}(s_1,s_2,s_3,s_4)|^2 &= \frac{e^4}{N}\left|-\frac{a_1}{t}+\frac{a_2}{s}\right|^2\\
&= \frac{e^4}{N}\left(-\frac{a_1}{t}+\frac{a_2}{s}\right)\left(-\frac{a_1}{t}+\frac{a_2}{s}\right)^*\\
&= \frac{e^4}{N}\left(\frac{a_1a_1^*}{t^2}-\frac{a_1a_2^*}{st}-\frac{a_1^*a_2}{st}+\frac{a_2a_2^*}{s^2}\right)
\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}|\mathcal{M}(s_1,s_2,s_3,s_4)|^2\\
&= \frac{e^4}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}\frac{1}{N}\left(\frac{a_1a_1^*}{t^2}-\frac{a_1a_2^*}{st}-\frac{a_1^*a_2}{st}+\frac{a_2a_2^*}{s^2}\right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned}
f_{11} &= \frac{1}{N}\sum_{\text{spins}} a_1a_1^* = \operatorname{Tr}\Big((\not p_1 - m)\gamma^\mu(\not p_3 - m)\gamma^\nu\Big)\operatorname{Tr}\Big((\not p_4 + m)\gamma_\mu(\not p_2 + m)\gamma_\nu\Big)\\
f_{12} &= \frac{1}{N}\sum_{\text{spins}} a_1a_2^* = \operatorname{Tr}\Big((\not p_1 - m)\gamma^\mu(\not p_2 + m)\gamma^\nu(\not p_4 + m)\gamma_\mu(\not p_3 - m)\gamma_\nu\Big)\\
f_{22} &= \frac{1}{N}\sum_{\text{spins}} a_2a_2^* = \operatorname{Tr}\Big((\not p_1 - m)\gamma^\mu(\not p_2 + m)\gamma^\nu\Big)\operatorname{Tr}\Big((\not p_4 + m)\gamma_\mu(\not p_3 - m)\gamma_\nu\Big)
\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{t^2}-\frac{f_{12}}{st}-\frac{f_{12}^*}{st}+\frac{f_{22}}{s^2}\right)$$

Run “bhabha-scattering-1.txt” to verify the Casimir trick.

These formulas compute probability densities from dot products.

$$
\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) - 32m^2(p_1 \cdot p_3) - 32m^2(p_2 \cdot p_4) + 64m^4 \\
f_{12} &= -32(p_1 \cdot p_4)(p_2 \cdot p_3) - 16m^2(p_1 \cdot p_2) + 16m^2(p_1 \cdot p_3) - 16m^2(p_1 \cdot p_4) \\
&\qquad - 16m^2(p_2 \cdot p_3) + 16m^2(p_2 \cdot p_4) - 16m^2(p_3 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_3)(p_2 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) + 32m^2(p_1 \cdot p_2) + 32m^2(p_3 \cdot p_4) + 64m^4
\end{aligned}
$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ the formulas are

$$
\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= -8u^2 + 64um^2 - 96m^4 \\
f_{22} &= 8t^2 + 8u^2 - 64tm^2 - 64um^2 + 192m^4
\end{aligned}
$$

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8s^2 + 8u^2 \\
f_{12} &= -8u^2 \\
f_{22} &= 8t^2 + 8u^2
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
s &= 4E^2 \\
t &= -2E^2(1 - \cos\theta) = -4E^2 \sin^2(\theta/2) \\
u &= -2E^2(1 + \cos\theta) = -4E^2 \cos^2(\theta/2)
\end{aligned}
$$

The corresponding expected probability density is

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} + \frac{16u^2}{st} + \frac{8t^2 + 8u^2}{s^2} \right) \\
&= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2 + u^2}{s^2} \right) \\
&= 2e^4 \left(\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1 + \cos^2\theta}{2} \right)
\end{aligned}
$$

Run “bhabha-scattering-2.txt” to verify.

This is the differential cross section for Bhabha scattering.

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{\alpha^2}{8E^2}\left(\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1+\cos^2\theta}{2}\right)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function.

Let

$$I(\xi) = 2\pi\int_\alpha^\xi \frac{d\sigma}{d\Omega}\sin\theta\,d\theta, \quad \alpha \le \xi \le \pi$$

for some $\alpha > 0$. The support interval is restricted because $d\sigma$ is undefined for $\theta = 0$.

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)}{I(\pi)}, \quad \alpha \le \theta \le \pi$$

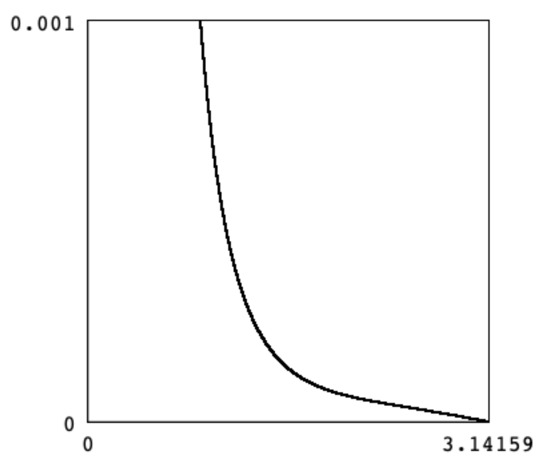
Hence

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{\sin\theta}{I(\pi)}\left(\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1+\cos^2\theta}{2}\right), \quad \alpha \le \theta \le \pi$$

Run "bhabha-scattering-3.txt" to draw $f(\theta)$ for $\alpha = \pi/180$.



Here is a probability distribution for 45° bins with $\alpha = 45°$.

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	45°	–
45°	90°	0.83
90°	135°	0.13
135°	180°	0.04

The following Bhabha scattering data is adapted from SLAC-PUB-1501.

	Bin	$\cos\theta$ (interval)	Count
(Smallest θ)	1	$0.6, 0.5$	4432
	2	$0.5, 0.4$	2841
	3	$0.4, 0.3$	2045
	4	$0.3, 0.2$	1420
	5	$0.2, 0.1$	1136
	6	$0.1, 0.0$	852
	7	$0.0, -0.1$	656
	8	$-0.1, -0.2$	625
	9	$-0.2, -0.3$	511
	10	$-0.3, -0.4$	455
	11	$-0.4, -0.5$	402
(Largest θ)	12	$-0.5, -0.6$	398

"Count" is the number of Bhabha scattering events observed per bin. Let us see if the density function $\langle|\mathcal{M}|^2\rangle$ explains the distribution of counts in the table. Start by integrating $\langle|\mathcal{M}|^2\rangle$ over all the bins to obtain a normalization constant.

$$\int_{\text{bins}} \langle|\mathcal{M}|^2\rangle \, d\Omega = \int_0^{2\pi} \int_{\arccos 0.6}^{\arccos -0.6} \langle|\mathcal{M}|^2\rangle \sin\theta \, d\theta \, d\phi = 2\pi \times 9.3817 \times 2e^4$$

Let

$$f(\theta) = \frac{\langle|\mathcal{M}|^2\rangle}{2\pi \times 9.3817 \times 2e^4} = \frac{1}{2\pi \times 9.3817} \left(\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1+\cos^2\theta}{2} \right)$$

The probability of a scattering event occurring in an interval θ_1 to θ_2 is obtained by integrating $f(\theta)$ over that interval.

$$P(\theta_1 < \theta < \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} f(\theta) \sin\theta \, d\theta \, d\phi = 2\pi \int_{\theta_1}^{\theta_2} f(\theta) \sin\theta \, d\theta$$

The total number of counts in the table is 15773. To obtain a predicted distribution, multiply 15773 times the probability for each bin. For example, for the first bin we have

$$P(\arccos 0.6 < \theta < \arccos 0.5) \times 15773 = 4598$$

Repeat for all bins to obtain the following predicted distribution.

Bin	$\cos\theta$ (interval)	Count	Predicted
1	$0.6, 0.5$	4432	4598
2	$0.5, 0.4$	2841	2880
3	$0.4, 0.3$	2045	1955
4	$0.3, 0.2$	1420	1410
5	$0.2, 0.1$	1136	1068
6	$0.1, 0.0$	852	843
7	$0.0, -0.1$	656	689
8	$-0.1, -0.2$	625	582
9	$-0.2, -0.3$	511	505
10	$-0.3, -0.4$	455	450
11	$-0.4, -0.5$	402	411
12	$-0.5, -0.6$	398	382

The coefficient of determination R^2 measures how well predicted values fit the real data. Let y be observed counts per bin and let $\hat{y}$ be predicted counts per bin. Then

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.997$$

The result indicates that the model $\langle|\mathcal{M}|^2\rangle$ explains 99.7% of the variance in the data.

Run "bhabha-scattering-4.txt" to verify.

The following table shows DESY-PETRA Bhabha scattering data obtained from HEP Data.[1]

x	y
-0.73	0.10115
-0.6495	0.12235
-0.5495	0.11258
-0.4494	0.09968
-0.3493	0.14749
-0.2491	0.14017
-0.149	0.1819
-0.0488	0.22964
0.0514	0.25312
0.1516	0.30998
0.252	0.40898
0.3524	0.62695
0.4529	0.91803
0.5537	1.51743
0.6548	2.56714
0.7323	4.30279

Data x and y have the following relationship with the cross section model.

$$x = \cos\theta \qquad y = \frac{d\sigma}{d\Omega}$$

The differential cross section for Bhabha scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2+u^2}{s^2}\right)$$

The predicted cross section $\hat{y}$ is computed from data x and beam energy E as

$$\hat{y} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2+u^2}{s^2}\right) \times (\hbar c)^2 \times 10^{37}$$

where

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1-x) \\ u &= -2E^2(1+x) \end{aligned}$$

Factor $(\hbar c)^2$ converts the result to SI and factor 10^{37} converts square meters to nanobarns.

The following table shows $\hat{y}$ for $E = 7.0\,\text{GeV}$.

[1] `www.hepdata.net/record/ins191231` (Table 3, 14.0 GeV)

x	y	$\hat{y}$
−0.73	0.10115	0.110296
−0.6495	0.12235	0.113816
−0.5495	0.11258	0.120101
−0.4494	0.09968	0.129075
−0.3493	0.14749	0.141592
−0.2491	0.14017	0.158934
−0.149	0.1819	0.182976
−0.0488	0.22964	0.216737
0.0514	0.25312	0.264989
0.1516	0.30998	0.335782
0.252	0.40898	0.44363
0.3524	0.62695	0.615528
0.4529	0.91803	0.9077
0.5537	1.51743	1.45175
0.6548	2.56714	2.60928
0.7323	4.30279	4.61509

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.995$$

The result indicates that the model $d\sigma$ explains 99.5% of the variance in the data.

Run “bhabha-scattering-5.txt” to verify.

Here are a few notes about how the scripts work. In component notation the trace operators of the Casimir trick become sums over the repeated index α.

$$
\begin{aligned}
f_{11} &= \left((\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right) \left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right) \\
f_{12} &= (\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\
f_{22} &= \left((\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right) \left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)
\end{aligned}
$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho \quad \rightarrow$ `gammaT = transpose(gamma)`

$\gamma^\beta{}_{\mu\rho} \quad \rightarrow$ `gammaL = transpose(dot(gmunu,gamma))`

Define the following 4×4 matrices.

$(\not{p}_1 - m) \quad \rightarrow$ `X1 = pslash1 - m I`

$(\not{p}_2 + m) \quad \rightarrow$ `X2 = pslash2 + m I`

$(\not{p}_3 - m) \quad \rightarrow$ `X3 = pslash3 - m I`

$(\not{p}_4 + m) \quad \rightarrow$ `X4 = pslash4 + m I`

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow$ `T1 = contract(dot(X1,gammaT,X3,gammaT),1,4)`

$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow$ `T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)`

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$f_{11} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow$ `f11 = contract(dot(T1,transpose(T2)))`

Follow suit for f_{22}.

$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow$ `T1 = contract(dot(X1,gammaT,X2,gammaT),1,4)`

$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow$ `T2 = contract(dot(X4,gammaL,X3,gammaL),1,4)`

Hence

$f_{22} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow$ `f22 = contract(dot(T1,transpose(T2)))`

The calculation of f_{12} begins with

$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$

$\rightarrow$ `T = contract(dot(X1,gammaT,X2,gammaT,X4,gammaL,X3,gammaL),1,6)`

Then sum over repeated indices μ and ν.

$f_{12} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow$ `f12 = contract(contract(T,1,3))`